Why Tau Should be Used in Place of Pi as the Circle Constant

Unit 3 Project Part 2: Tau vs Pi

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1 Introduction

I am assuming everyone reading this is knowledgeable in trigonometry, but I will give a quick introduction to what a circle constant is so I can construct my argument on tau.

The most common circle constant is defined as $\pi = \frac{C}{d}$ where C is the circumference of a circle and d is its diameter. A circle constant relating a circle's circumference to its width is very useful, as one could imagine. It can be used to calculate how far a wheel travels or the rotations of a motor. It also has applications in mathematics, such as the trigonometric functions which relate to the unit circle, and countless other fields of mathematics.

A known circle constant has existed much before the common era, possibly as far back as 1900 BCE.[1] By the 18th century, the circle constant we know as π was calculated to 71 decimal places. The earliest known use of π alone to represent a circle constant was in 1709 by William Jones. It was used to represent the "1/2 Periphery." Euler's first use of π was in 1727 where he set $\pi = 6.28\ldots$. It wasn't until 1736 where he first used π to represent half of a rotation around a circle, and it was still ambiguous for decades afterwards.

But, there is one fundamental flaw that underlies the usage of π. This can be demonstrated using the unit circle. It's called the unit circle for a reason; it has a radius of 1 unit. This means when calculating the circumference with π, you must actually consider the number 2π because the diameter is 2. This one simple feature leads to many, many things in mathematics that must carry around a needless factor of 2 or $\frac{1}{2}$.

This is the core of what tau seeks to rectify. From the time π was first used as a symbol in this context, it was ambiguous whether π should equal $\frac{C}{d}$ or $\frac{C}{r}$[2]. Perhaps, if it had gone the other way, there would be less headaches for students learning, and more purity in the many, many fields π is relevant in. Obviously, for this paper using the symbol π to represent $\frac{C}{r}$ would lead to even more confusion, so an alternative that is common in the mathematical community is $\tau = \frac{C}{r} = 2\pi \approx 6.28$.

2 τ in Trigonometry

π simply shows up everywhere in mathematics. It exists in formulas you would expect it to, like area of a circle or sphere. But it also shows up in tons of formulas in physics that seem to have nothing to do with circles, or the solutions to many integrals and infinite series that seemingly have no circles in sight. In many of these cases, π appears accompanied by a factor of 2, but in others it doesn't, which I find far more interesting. It's easy to look at all the places where 2π shows up and use that as the only piece of evidence in support for τ, and I will do that, but searching for the hidden circle in seemingly random equations really makes τ shine.

2.1 τ in the Unit Circle

One of the areas of math that most clearly shows the effectiveness of τ is trigonometry[3]. As a first impression, trigonometry seems to be just about triangles, but with the use of the unit circle, it becomes so much more. By placing a right triangle in a circle with radius 1 centered on the origin such that the right angle's point sits at the point (1, 0) and another vertex is on the origin, trigonometry is extended beyond right triangles.

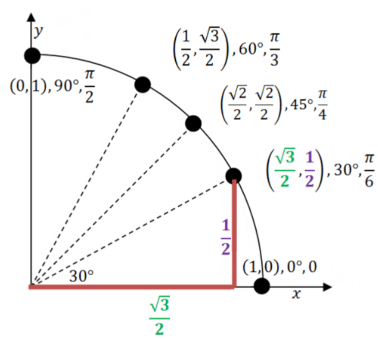

Figure 1: the unit circle [4]

This image shows how the familiar trigonometric ratios are seen in the context of the unit circle. This diagram also shows angle values in radians. Radians are a useful way to measure angles, as they directly show what fraction of a full turn the angle is in terms of the radius of the circle, or at least it should. The diagram shows a quarter turn as $\frac{\pi}{2}$ radians. Wouldn't it make more sense for $\frac{1}{4}$ of a circle to be $\frac{\pi}{4}$ radians? Well, That's where τ comes in.

The fact that a quarter of a turn is $\frac{\pi}{2}$ radians comes from $\pi = \frac{C}{d}$. Diameter is $2r$ so $\pi = \frac{C}{2r}$ On a unit circle where the radius is 1 that is simply $\frac{C}{2} = \pi$. This means that π is equal to half the circumference. Half of that means a quarter of the circumference is $\frac{\pi}{2}$ rads. The factor of 2 introduced by diameter is what is throwing things off.

Using the formula $\tau = \frac{C}{r}$ yields $\tau = C$. This shouldn't be a mind blowing fact, this should be obvious. Of course the circumference of the unit circle should be equal to this constant because that is what defines it. It seems illogical for it to be any other way. Now, using radians seems so much simpler. If I saw $\frac{7\pi}{18}$ I would freak out. But $\frac{7\tau}{9}$ clearly conveys the idea of seven ninths of a circle. This clarity doesn't stop at the unit circle, it extends to every identity and idea in trigonometry, and beyond.

2.2 τ in Trigonometric Identities

Using τ in trigonometric identities is similarly easier to understand. Take the periodicity of sine:

$$\sin\theta = \sin(\theta + 2\pi)$$

For someone learning about this for the first time, it is unnecessarily confusing. A student may rightfully ask "Why is there a 2 here if the angle is only 1 more rotation" If you understand that 2π means one rotation, the identity makes sense, but that brings it back to "Why does 2π represent 1 rotation?"

Using tau painlessly conveys the notion of 1 rotation with 1 symbol:

$$\sin\theta = \sin(\theta + \tau)$$

Another identity that uses the circle constant is

$$\sin\theta = \cos(\frac{\pi}{2} - \theta)$$

This equation doesn't have a 2π that can easily be explained away. Here is a proof of the identity using τ.

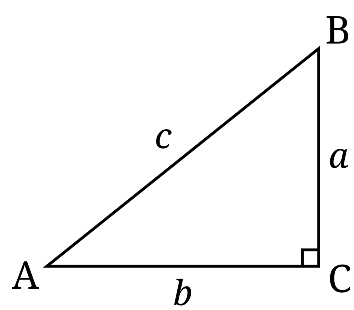

Figure 2: right triangle[5]

Proof.

$$\sin A = \frac{a}{c}$$
$$\cos B = \frac{a}{c}$$

The sum of the angles in the triangle is equal to a half of a rotation by the angle sum formula. Therefore,

$$\begin{aligned} A + B + C &= \frac{\tau}{2} \\ C &= \frac{\tau}{4} \\ B &= \frac{\tau}{4} - A \\ \sin A = \cos B &= \cos(\frac{\tau}{4} - A) \end{aligned}$$

□

This is an example of something that doesn't really change. The proof or identity isn't drastically different, but it does give an insight into another way of looking at it. If $\frac{\tau}{4}$ is a quarter of a turn, we can visualize this by looking at the unit circle.

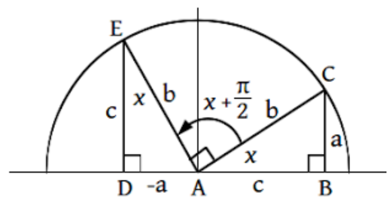

Figure 3: cofunction identity[6]

Figure 3 uses the unit circle to prove the cofunction identity. Even though it uses π, one can see how this is true geometrically.

2.3 τ in Trigonometric Function Graphs

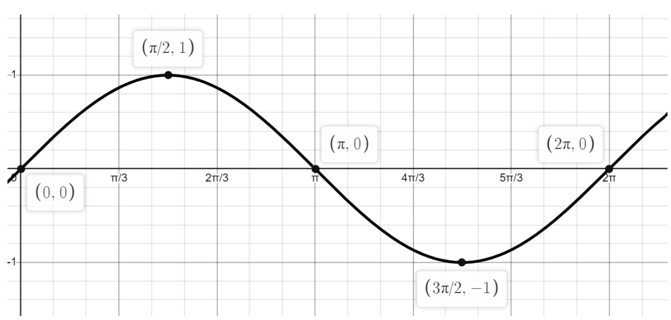

Figure 4: desmos.com

Figure 4 shows a sine wave, its zeros and its inflection points shown with π as the circle constant. From the definition of the function as the y-value of a point moving along the edge of a circle, the connection can be made that the point $(2\pi, 0)$ represents how one rotation around starting at $(1, 0)$ gets you back where you started. Again, notice the dichotomy between the point $(2\pi, 0)$ and one rotation. The use of τ simply shows the connection more clearly. The sine function repeats after τ radians because one rotation around a circle is τ radians.

3 τ in Education

In the previous section, I explained just some of the ways in which the use of τ makes trigonometry more intuitive and the connections that are made more obvious. This makes τ the obvious choice for school curriculums to use to teach students about mathematics.

For me, trigonometry was this thing that was hinted at throughout my math courses leading up to it. We were taught about π as this magical number that will be extremely useful in more advanced math, sort of how e is treated in my precalculus course now. We were told π is the circumference of any circle divided by its diameter and told how to calculate the area and perimeter of a circle. At this level, it does seem trivial to reconsider the circle constant used. In the case of $A = \pi r^2$, replacing it with $\frac{\tau}{2}$ even seems to complicate things further. But τ isn't just about making computations easier, because, yes, it is just a factor of 2. τ is about making math more intuitive and approachable. It gives a window into the mechanisms of trigonometry that is obscured by the use of π.

But even with all the advantages that τ provides, it seems almost impossible to switch now. For almost 300 years, anyone with a high school education learned about trigonometry with π. It would be crazy to go back through every textbook and lesson and rewrite it to use τ. But, at least for the American education experience there is an opportunity. The College Board is planning to create a new Advanced Placement class called AP Precalculus. This would be offered in high schools nationwide starting in the 2023-24 school year. According to College Board, "AP Precalculus prepares students for other higher-level mathematics and science courses."[7] Most of the course is designed around working with different kinds of functions, and a major part is trigonometric functions.

This course would be an excellent place to introduce students to τ. There are even some textbooks written only using τ as its circle constant[8][9]. Realistically, students would have to learn with both π and τ as the overwhelming majority of learning resources and applications use π, but it could be the start of a global movement to replace π with a more intuitive and applicable way of expressing the relationship between a circle's circumference and radius.

References

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