Douglass Final Project

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1 Proof $\sqrt{2}$ is irrational

Proof. Suppose $\sqrt{2}$ is irrational. By the definition of rational numbers, $\sqrt{2}$ can be expressed as $\frac{p}{q}$ where $p, q \in (Z)$ and have no common factors.

$$\sqrt{2} = \frac{p}{q}$$

From this we get

$$2 = \left(\frac{p}{q}\right)^2$$
$$2 = \frac{p^2}{q^2}$$
$$p^2 = 2q^2$$

Because p and q are both integers, we know p^2 is an even number. If a square is even, its square root must also be even, so we know p is even. This means p can be represented by $2n$, so $p^2 = (2n)^2 = 4n^2 \therefore 4$ is a factor of p. Now we can rewrite (4) as

$$4n^2 = 2q^2$$
$$2n^2 = q^2$$

From this we know that q^2 is even, and using similar logic that we used with p, q is even. Since we have showed that p and q are both even and therefore divisible by 2, our assumption of p and q having no common factors is both false and true, meaning there is a contradiction. This means our original assumption that $\sqrt{2}$ is rational must be false, meaning $\sqrt{2}$ must be rational. □

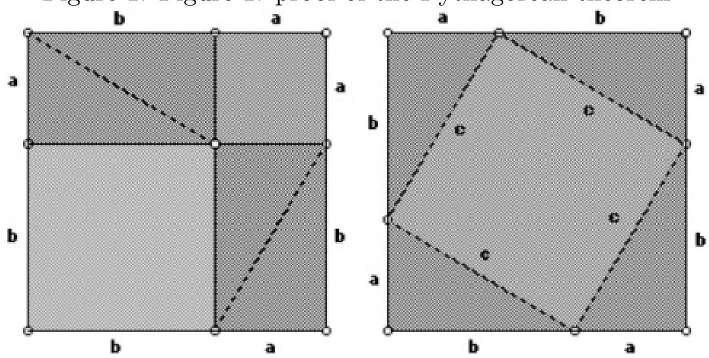
2 Proof of the Pythagorean Theorem

The Pythagorean is as follows:

$$a^2 + b^2 = c^2$$

where a and b are the legs of a right triangle and c is the hypotenuse.

Figure 1: Figure 1: proof of the Pythagorean theorem



3 Proofs of the Gaussian integral

The Gaussian integral is as follows:

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

3.1 Method 1: Polar coordinates

First, we set

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

as we will need to manipulate I as you will soon see. Next, square both sides of the equation:

$$\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) e^{-y^2} dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dxdy \\
&= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2} r d\theta dr \\
&= \int_{0}^{\infty} e^{-r^2} r\theta\Big|_{0}^{\theta=2\pi} dr \\
&= \int_{0}^{\infty} re^{-r^2}(2\pi) - re^{-r^2}(0) dr \\
&= \int_{0}^{\infty} 2\pi re^{-r^2} dr
\end{aligned}$$

let $u = -r^2, du = -2rdr, dr = \frac{du}{-2r}$

$$\begin{aligned}
I^2 &= \int_{0}^{-\infty} 2\pi re^{u} \frac{du}{-2r} \\
&= -\int_{0}^{-\infty} \pi e^{u} du \\
&= \pi\int_{-\infty}^{0} e^{u} du \\
&= \pi e^{u}\Big|_{-\infty}^{0} \\
&= \pi e^{0} - \pi \lim_{u\to-\infty} e^{u} \\
&= \pi - \pi(0) \\
&= \pi \\
I &= \sqrt{\pi}
\end{aligned}$$

3.2 Method 2: Differentiation under the integral sign

When using this method to solve this integral, we set

$$I(b) = \left(\int_0^b e^{-x^2} dx \right)^2$$

and because e^{-x^2} is an even function, $2\sqrt{\lim_{b\to\infty} I(b)}$ will be the solution. Now, differentiate both sides with respect to t:

$$\begin{aligned} I'(b) &= 2\int_0^b e^{-x^2} dx \cdot e^{-b^2} \\ &= 2e^{-b^2} \int_0^b e^{-x^2} dx \end{aligned}$$

let $x = by, y = \frac{x}{b}, dx = bdy$

$$\begin{aligned} I'(b) &= 2e^{-b^2} \int_0^1 be^{-b^2y^2} dy \\ &= \int_0^1 2be^{-(1+y^2)b^2} dy \\ &= \int_0^1 -\frac{\partial}{\partial b} \frac{e^{-(1+y^2)b^2}}{1+y^2} dy \\ &= -\frac{d}{db} \int_0^1 \frac{e^{-(1+y^2)b^2}}{1+y^2} dy \end{aligned}$$

Now let

$$\begin{aligned} J(b) &= \int_0^1 \frac{e^{-b^2(1+x^2)}}{1+x^2} dx \\ J'(b) &= \frac{d}{db} \int_0^1 \frac{e^{-b^2(1+x^2)}}{1+x^2} dx \end{aligned}$$

Notice,

$$\begin{aligned} I'(b) &= -J'(b) \\ I(b) &= -J(b) + c \end{aligned}$$

Now, see what happens as $b \to 0$

$$\begin{aligned}
I(0) &= -J(0) + c \\
\left(\int_0^0 e^{-x^2} dx\right)^2 &= \int_0^1 \frac{e^{-0^2(1+x^2)}}{1+x^2} dx + c \\
0 &= \int_0^1 \frac{1}{1+x^2} dx + c \\
c &= \arctan x \Big|_0^1 \\
c &= \frac{\pi}{4} \\
I(b) = \frac{\pi}{4} - J(b)
\end{aligned}$$

Now, let $b \to \infty$

$$\begin{aligned}
\left(\int_0^\infty e^{-x^2} dx\right)^2 &= \frac{\pi}{4} - \int_0^1 \frac{e^{-\infty^2(1+x^2)}}{1+x^2} dx \\
&= \frac{\pi}{4} - \int_0^1 0 dx \\
&= \frac{\pi}{4} \\
\int_0^\infty e^{-x^2} dx &= \sqrt{\frac{\pi}{4}} \\
&= \frac{\sqrt{\pi}}{2}
\end{aligned}$$

So the answer to our original integral is $\sqrt{\pi}$.

4 Reflection

For this project, I compiled a few of my favorite proofs from areas of mathematics to convey the argument that math isn't just cold, soulless facts, math can be like telling a story or a way to make an argument. I selected these proofs specifically to back up my argument. For a lot of people, math can seem like an academic barrier, and if they aren't doing well they may give up on doing well in school. Many people see math as an obstacle in school or something that is used to find solutions to problems in the "real world," but I see math as interesting in and of itself. The first proof I presented was the proof that 2 is irrational. This is an example of a short and relatively simple proof that can be easily understood without much experience in math. This is commonly used to introduce people to the general form of a proof. But, many people are not introduced to this kind of thinking in their education or at all. The fact that I am exposed to these interesting topics in high school, or just that I can even go to high school is not universal. For example, people like Fredrick Douglass did not have a formal education. Douglass had to learn to read from orphans in the streets of Baltimore.

I also used the proof of the pythagorean theorem to back up my argument. The pythagorean is, possibly infamously, known to almost anyone who has a high school education. Many people may have forgotten what it is used to calculate, but a2 + b2 = c2 is one of those equations that people associate with math in general. For me, this was one of the first times I enjoyed math. This equation was seemingly so simple, but it describes something so fundamental. When I learned about this I remember thinking "why?" The proof I included in my project is the most beautiful proof I could find. The proof is so simple and elegant that it can be described in a single picture. This was kind of a 'leaving the cave' moment, like in Plato's Allegory of the Cave. Now, whenever I learn about a new theorem in math or another fact in another field, I always want to know why it's true, or how I could have come up with it myself. There are also many other times, like when learning about the later proofs, that my jaw non-metaphorically dropped.

Next, I looked at two very different proofs of something called the gaussian integral. This is a problem from multivariable calculus that is famous for having many different proofs that use vastly different areas of mathematics to solve. I chose the two that I could understand the best. I won't go into the details, but one method converts the integral from cartesian coordinates to polar coordinates, and the other uses a technique called differentiation under the integral sign. These are examples of how there are always multiple ways to solve the problem and, contrary to most people's beliefs about math, there is not always one answer. Well, in this case there is one right conclusion that can be reached by many different answers. In a way, this represents freedom that a lot of people don't see when they think of math. This can relate back to the allegory of the cave, in that once you see math as something creative, or a way to make an argument, it can be hard for other people to understand.

The medium I chose to convey my argument was a paper written in LaTeX, a text formatting language used to write math digitally. I chose this medium because it is how most math papers are written, and it makes it really easy to convey my ideas. Lately, I have been trying to learn how to use it better, so it was good practice. I feel that it was easier to convey my ideas than if I had used another medium, like the equations feature in google docs or writing it on a poster board. It also makes it look more professional, which could be an appeal to ethos. I am largely making a definition or a value claim, that math can have creativity and proofs can be like telling a story. I would like this to be graded at the honors level.