6.7 Inverses and Determinants

Nathan Birkett

January 2023

1 Question 41

Use the definition to evaluate the determinant of $A := \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 3 & -1 \end{bmatrix}$

1.1 row 1

$$\begin{aligned}
\det A &= 2\det\begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} - 1\det\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} + 1\det\begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix} \\
&= 2(0\cdot 1 - 2\cdot 3) - (-1\cdot -1 - 2\cdot 1) + (-1\cdot 3 - 0\cdot 1) \\
&= -12 + 1 - 3 \\
&= -14
\end{aligned}$$

1.2 row 2

$$\begin{aligned}
\det A &= - - 1\det\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} + 0\det\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \\
&= (1\cdot -1 - 1\cdot 3) - 2(2\cdot 3 - 1\cdot 1) \\
&= -4 - 10 \\
&= -14
\end{aligned}$$

2 Question 55

Prove that the inverse of the matrix $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Proof. Create the augmented matrix $(A|I)$ and using Gaussian elimination, reduce A to I. A^{-1} will be the right side.

$$\frac{R_1}{a} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$R_2 - cR_1 \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\frac{R_2}{d-\frac{bc}{a}} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d-\frac{bc}{a} & \frac{-c}{a} & 1 \end{array}\right]$$

$$R_1 - \frac{b}{a}R_2 \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{a(d-\frac{bc}{a})} & \frac{1}{d-\frac{bc}{a}} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} - \frac{bc}{a^2(d-\frac{bc}{a})} & \frac{-b}{a(d-\frac{bc}{a})} \\ 0 & 1 & \frac{-c}{a(d-\frac{bc}{a})} & \frac{1}{d-\frac{bc}{a}} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{ad-bc+bc}{a^2d-abc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

□

3 Question 69

Let $A := [a_{ij}]$ be an $n \times n$ matrix.

3.1 a)

Prove that if every element of a row or column of A is multiplied by the real number c, then the determinant of A is multiplied by c.

Proof. Let A^* be A where a row or column is multiplied by c. Now, look at the Leibniz formula for the determinant:

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}.$$

The definitions of σ and S_n etc. are unimportant for this proof; what's important is that multiplying a column by c gives

$$\det A^* = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) c \prod_{i=1}^{n} a_{i,\sigma_i} = c \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i} = c \det A$$

as the product iterates through every row so exactly 1 of the factors in the product will be multiplied by c. To prove this for rows, recall that the transpose preserves the determinant, which is shown by swapping the coordinates of each element in the matrix:

$$\det A = \det A^\top = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma_i,i}.$$

Similar logic can now be used, as the rows in $A^\top$ represent the columns in A, so

$$\det A^* = \det A^{*\top} = c \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma_i,i} = c \det A^\top = c \det A.$$

□

3.2 b)

Prove that if all the entries above the main diagonal (or all below it) of a matrix are zero, the determinant is the product of the elements on the main diagonal.

Proof. To prove this, we must look into the notation I used in the last proof. σ represents a permutation of the set $\{1,2,\ldots,n\}$, and the set of these σs is denoted by S_n. σ_i is the i-th element of σ and $\operatorname{sgn}(\sigma)$ still isn't important.[1]

[1] If you're really curious, $\operatorname{sgn}(\sigma)$ is the signature, or parity, of σ. It is 1 if it takes an even number of swaps to reach the permutation, and -1 if it's odd.

Now consider $\sigma \in S_n = \{1, 2, \ldots, n\}$, that is, nothing has been done to the set. The term represented in the Leibniz formula is

$$\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i} = \prod_{i=1}^{n} a_{i,i}$$

because the i^{th} element of {1,2,...,n} is i and $\operatorname{sgn}(\sigma) = 1$. You'll notice this is the product of the diagonals of A. This is the only term where this is true as I will demonstrate.

Say you have some element $a_{a,b}$. If $a < b$, $a_{a,b} = 0$ because any elements above the diagonal are 0. If $b < a$, $a_{a,b}$ will be below the main diagonal which can be nonzero. But, recall that a permutation can be represented by some amount of swapping of 2 elements. If 2 elements are swapped in σ, the corresponding factor is $a_{b,a}$ instead of $a_{a,b}$, meaning if $a \neq b$, one of the 2 possible elements will be 0. Consider the permutation where the first two elements are swapped: {2,1,3,4,...,n}. The corresponding term will look like:

$$-1 \cdot a_{1,2} \cdot a_{2,1} \cdot a_{3,3} \cdots = -1 \cdot 0 \cdot a_{2,1} \cdot a_{3,3} \cdots = 0.$$

Because at least 1 factor is guaranteed to be 0, the whole sum will be 0 except for $\sigma = \{1, 2, \ldots, n\}$, who's corresponding term is exactly the product of the main diagonal. It can be seen that this is also the case if all the elements below the diagonal are 0, because that would be the transpose of a matrix with zeroes above the diagonal. $\square$